

A connection formula of a divergent bilateral basic hypergeometric function

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Abstract

We give the new connection formula for the divergent bilateral basic hypergeometric series ${}_2\psi_2(a_1, a_2; b_1; q, x)$ by the using of the q -Borel-Laplace resummation method and Slater's formula. The connection coefficients are given by elliptic functions.

1 Introduction

In this paper, we show a connection formula for a divergent *bilateral* basic hypergeometric function

$${}_2\psi_1(a_1, a_2; b_1; q, x) := \sum_{n \in \mathbb{Z}} \frac{(a_1; q)_n (a_2; q)_n}{(b_1; q)_n (q; q)_n} \left\{ (-1)^n q^{\frac{n(n-1)}{2}} \right\}^{-1} x^n. \quad (1)$$

Here, $(a; q)_n, (n \in \mathbb{Z})$ is the q -shifted factorial (see [3] for more details). We assume that $q \in \mathbb{C}^*$ satisfies $0 < |q| < 1$. The function (1) satisfies the second order linear q -difference equation

$$\left(\frac{b_1}{q^2} - a_1 a_2 x \right) u(q^2 x) - \left\{ \frac{1}{q} - (a_1 + a_2)x \right\} u(qx) - xu(x) = 0. \quad (2)$$

The equation (2) also has the *unilateral* solutions around infinity:

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$$v_1(x) = \frac{\theta(a_1x)}{\theta(x)} \sum_{n \geq 0} \frac{(qa_1/b_1; q)_n (b_1/a_1 a_2 x)^n}{(qa_1/a_2; q)_n (q; q)_n},$$

$$v_2(x) = \frac{\theta(a_2x)}{\theta(x)} \sum_{n \geq 0} \frac{(qa_2/b_1; q)_n (b_1/a_1 a_2 x)^n}{(qa_2/a_1; q)_n (q; q)_n},$$

provided that the function $\theta(x) := \sum_{n \in \mathbb{Z}} q^{n(n-1)/2} x^n$, $\forall x \in \mathbb{C}^*$ is the theta function of Jacobi. The aim of this paper is to give the connection formula between $v_1(x)$, $v_2(x)$ and the divergent series (1) as follows:

Theorem. For any $x \in \mathbb{C}^* \setminus -\lambda q^{\mathbb{Z}}$, we have

$$\begin{aligned} & (\mathcal{L}_{q,\lambda}^+ \circ \mathcal{B}_q^+ \psi_1(a_1, a_2; b_1; q, x))(x) \\ &= \frac{(1/a_2, qa_1/a_2, b_1/a_1, q; q)_\infty}{(b_1, q/a_1, a_1/a_2, qa_2/a_1; q)_\infty} \frac{\theta(a_1\lambda/q)}{\theta(\lambda/q)} \frac{\theta(a_1qx/\lambda)}{\theta(qx/\lambda)} \frac{\theta(x)}{\theta(a_1x)} v_1(x) \\ &+ \frac{(1/a_1, qa_2/a_1, b_1/a_2, q; q)_\infty}{(b_1, q/a_2, a_2/a_1, qa_1/a_2; q)_\infty} \frac{\theta(a_2\lambda/q)}{\theta(\lambda/q)} \frac{\theta(a_2qx/\lambda)}{\theta(qx/\lambda)} \frac{\theta(x)}{\theta(a_2x)} v_2(x). \end{aligned}$$

Here, \mathcal{B}_q^+ and $\mathcal{L}_{q,\lambda}^+$ are the q -Borel-Laplace transformations (see section two). We remark that the q -elliptic functions (with the new parameter λ) appear in the connection coefficients.

At first, we review the connection problems on the linear q -difference equations. Connection problems on the linear q -difference equations with regular singular points were studied by G. D. Birkhoff [1] in 1914. Connection formulae for the second order linear q -difference equations are given by the following matrix form:

$$\begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix} = \begin{pmatrix} C_{11}(x) & C_{12}(x) \\ C_{21}(x) & C_{22}(x) \end{pmatrix} \begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix}.$$

The pair $(u_1(x), u_2(x))$ is a fundamental system of (unilateral) solutions around the origin and the pair $(v_1(x), v_2(x))$ is a fundamental system of solutions around the infinity. The connection coefficients C_{jk} ($1 \leq j, k \leq 2$) are elliptic functions:

$$\sigma_q C_{jk}(x) = C_{jk}(x), \quad C_{jk}(e^{2\pi i} x) = C_{jk}(x),$$

namely, q -periodic and unique valued functions.

The first example of the connection formula was given by G. N. Watson [10] in 1910. Watson gave the connection formula for Heine's basic hypergeometric series

$${}_2\varphi_1(a, b; c; q, x) := \sum_{n \geq 0} \frac{(a, b; q)_n}{(c; q)_n (q; q)_n} x^n$$

around the origin and around the infinity [3, page 117]. Heine's ${}_2\varphi_1(a, b; c; q, x)$ satisfies the q -difference equation

$$[(c - abqx)\sigma_q^2 - \{(c + q) - (a + b)qx\}\sigma_q + q(1 - x)]u(x) = 0. \quad (3)$$

The equation (3) also has a fundamental system of solutions around the infinity:

$$y_\infty^{(a,b)}(x) = \frac{\theta(ax)}{\theta(x)} {}_2\varphi_1\left(a, \frac{aq}{c}; \frac{aq}{b}; q, \frac{cq}{abx}\right)$$

and

$$y_\infty^{(b,a)}(x) = \frac{\theta(bx)}{\theta(x)} {}_2\varphi_1\left(b, \frac{bq}{c}; \frac{bq}{a}; q, \frac{cq}{abx}\right).$$

Watson's connection formula for ${}_2\varphi_1(a, b; c; q, x)$ is given by

$$\begin{aligned} {}_2\varphi_1(a, b; c; q; x) &= \frac{(b, c/a; q)_\infty \theta(-ax)_\infty}{(c, b/a; q)_\infty \theta(-x)_\infty} \frac{\theta(x)}{\theta(ax)} y_\infty^{(a,b)}(x) \\ &+ \frac{(a, c/b; q)_\infty \theta(-bx)_\infty}{(c, a/b; q)_\infty \theta(-x)_\infty} \frac{\theta(x)}{\theta(bx)} y_\infty^{(b,a)}(x). \end{aligned}$$

Here, connection coefficients are q -elliptic functions.

But connection formulae for q -difference equations with irregular singular points had not known for a long time. The irregularity of q -difference equations are studied using the Newton polygons by J.-P. Ramis, J. Sauloy and C. Zhang [8]. Recently, C. Zhang gave connection formulae for some confluent type basic hypergeometric series [12, 13]. Zhang also gives the connection formula for the divergent series ${}_2\varphi_0(a, b; -; q, x)$ in [11, 13] where he uses the q -Borel-Laplace transformations. In [5, 6], the author also gave the connection formula for the Hahn-Exton q -Bessel function and the q -confluent type function by the using of another kind of the q -Borel-Laplace transformations. These resummation methods are powerful tools for connection problems with irregular singular points.

In this paper, we apply the q -Borel-Laplace transformations for the bilateral series to the divergent bilateral basic hypergeometric series (1).

Definition 1. We assume that $f(x)$ is a formal power series $f(x) = \sum_{n \in \mathbb{Z}} a_n x^n$, $a_0 = 1$.

1. The q -Borel transformation is

$$(\mathcal{B}_q^+ f)(\xi) := \sum_{n \in \mathbb{Z}} a_n q^{\frac{n(n-1)}{2}} \xi^n (=:\psi(\xi)).$$

2. For any analytic function $\psi(\xi)$ around $\xi = 0$, the q -Laplace transformation is

$$(\mathcal{L}_{q,\lambda}^+ \psi)(x) := \frac{1}{1-q} \int_0^{\lambda\infty} \frac{\varphi(\xi)}{\theta_q\left(\frac{\xi}{x}\right)} \frac{d_q \xi}{\xi} = \sum_{n \in \mathbb{Z}} \frac{\varphi(\lambda q^n)}{\theta_q\left(\frac{\lambda q^n}{x}\right)}.$$

Here, this transformation is given by Jackson's q -integral [3, page 23].

The definition is a special case of one of the q -Laplace transformations in [2, 11]. The q -Borel transformation is the formal inverse of the q -Laplace transformation as follows:

Lemma 1 (Zhang, [11]). For any entire function $f(x)$, we have

$$\mathcal{L}_{q,\lambda}^+ \circ \mathcal{B}_q^+ f = f.$$

The applications of these transformations can be found in [13, 4]. We remark that these examples are connection formulae for the bilateral solution of the first order q -difference equations. But other formulae, especially more higher order and the degenerated (i.e., a confluent) case have not known. In the last section, we give the proof of the main theorem by the using of the q -Borel-Laplace transformations and Slater's formula [9].

2 Basic notations

In this section, we fix our notations. The q -shifted operator σ_q is given by $\sigma_q f(x) = f(qx)$. For any fixed $\lambda \in \mathbb{C}^* \setminus q^{\mathbb{Z}}$, the set $[\lambda; q]$ -spiral is $[\lambda; q] :=$

$\lambda q^{\mathbb{Z}} = \{\lambda q^k; k \in \mathbb{Z}\}$. The function $(a; q)_n$ is the q -shifted factorial;

$$(a; q)_n := \begin{cases} 1, & n = 0, \\ (1-a)(1-aq) \dots (1-aq^{n-1}), & n \geq 1, \\ [(1-aq^{-1})(1-aq^{-2}) \dots (1-aq^n)]^{-1}, & n \leq -1 \end{cases}$$

moreover, $(a; q)_\infty := \lim_{n \rightarrow \infty} (a; q)_n$ and

$$(a_1, a_2, \dots, a_m; q)_\infty := (a_1; q)_\infty (a_2; q)_\infty \dots (a_m; q)_\infty.$$

The basic hypergeometric series with the base q [3, page 4] is

$$\begin{aligned} {}_r\varphi_s(a_1, \dots, a_r; b_1, \dots, b_s; q, x) \\ := \sum_{n \geq 0} \frac{(a_1, \dots, a_r; q)_n}{(b_1, \dots, b_s; q)_n (q; q)_n} \left\{ (-1)^n q^{\frac{n(n-1)}{2}} \right\}^{1+s-r} x^n. \end{aligned}$$

The radius of convergence is ∞ , 1 or 0 according to whether $r-s < 1$, $r-s = 1$ or $r-s > 1$.

The bilateral basic hypergeometric series with the base q [3, page 137] is

$$\begin{aligned} {}_r\psi_s(a_1, \dots, a_r; b_1, \dots, b_s; q, x) \\ := \sum_{n \in \mathbb{Z}} \frac{(a_1, \dots, a_r; q)_n}{(b_1, \dots, b_s; q)_n} \left\{ (-1)^n q^{\frac{n(n-1)}{2}} \right\}^{s-r} x^n. \end{aligned}$$

The series ${}_r\psi_s(a_1, \dots, a_r; b_1, \dots, b_s; q, x)$ converges on:

$$\begin{aligned} r < s \quad & |x| > R := \left| \frac{b_1 b_2 \dots b_s}{a_1 a_2 \dots a_r} \right| \\ r = s \quad & R < |x| < 1 \\ s < r \quad & \text{divergent around the origin.} \end{aligned}$$

The theta function of Jacobi is important in connection problems. The theta function of Jacobi with the base q is

$$\theta_q(x) := \sum_{n \in \mathbb{Z}} q^{\frac{n(n-1)}{2}} x^n, \quad \forall x \in \mathbb{C}^*.$$

The theta function has the triple product identity

$$\theta_q(x) = \left(q, -x, -\frac{q}{x}; q \right)_\infty. \quad (4)$$

The theta function satisfies the first order q -difference equation $\theta_q(q^k x) = q^{-\frac{n(n-1)}{2}} x^{-k} \theta_q(x)$, $\forall k \in \mathbb{Z}$. The theta function also has the inversion formula $\theta_q(1/x) = \theta_q(x)/x$.

We remark that the function $\theta(-\lambda x)/\theta(\lambda x)$, $\forall \lambda \in \mathbb{C}^*$ satisfies a q -difference equation

$$u(qx) = -u(x),$$

which is also satisfied by the function $u(x) = e^{\pi i \left(\frac{\log x}{\log q} \right)}$.

3 Main theorem

In this section, we give the proof of the main theorem:

Theorem 1. *For any $x \in \mathbb{C}^* \setminus [-\lambda; q]$, we have*

$$\begin{aligned} & (\mathcal{L}_{q,\lambda}^+ \circ \mathcal{B}_q^+ {}_2\psi_1(a_1, a_2; b_1; q, x)) (x) \\ &= \frac{(1/a_2, qa_1/a_2, b_1/a_1, q; q)_\infty}{(b_1, q/a_1, a_1/a_2, qa_2/a_1; q)_\infty} \frac{\theta(a_1 \lambda/q)}{\theta(\lambda/q)} \frac{\theta(a_1 qx/\lambda)}{\theta(qx/\lambda)} \frac{\theta(x)}{\theta(a_1 x)} v_1(x) \\ &+ \frac{(1/a_1, qa_2/a_1, b_1/a_2, q; q)_\infty}{(b_1, q/a_2, a_2/a_1, qa_1/a_2; q)_\infty} \frac{\theta(a_2 \lambda/q)}{\theta(\lambda/q)} \frac{\theta(a_2 qx/\lambda)}{\theta(qx/\lambda)} \frac{\theta(x)}{\theta(a_2 x)} v_2(x). \end{aligned}$$

Here, $v_1(x)$ and $v_2(x)$ are a fundamental system of unilateral solutions (of equation (2)) around infinity:

$$\begin{aligned} v_1(x) &= \frac{\theta(a_1 x)}{\theta(x)} \sum_{n \geq 0} \frac{(qa_1/b_1; q)_n (b_1/a_1 a_2 x)^n}{(qa_1/a_2; q)_n (q; q)_n}, \\ v_2(x) &= \frac{\theta(a_2 x)}{\theta(x)} \sum_{n \geq 0} \frac{(qa_2/b_1; q)_n (b_1/a_1 a_2 x)^n}{(qa_2/a_1; q)_n (q; q)_n}. \end{aligned}$$

In the proof of main theorem, Slater's formula for the bilateral series[9] plays an important role. In subsection 3.1, we review Slater's formula for a bilateral basic hypergeometric series.

3.1 Slater's theorem

Slater gave the following connection formula between the bilateral series ${}_r\psi_r(a_1, \dots, a_r; b_1, \dots, b_r; q, x)$ around the origin and the basic hypergeometric function ${}_r\varphi_{r-1}$

Theorem 2 (Slater, [9]). *For any $|b_1 \cdots b_r/a_1 \cdots a_r| < |x| < 1$, we have*

$$\begin{aligned} & \frac{(b_1, \dots, b_r, q/a_1, \dots, q/a_r, x, q/x; q)_\infty}{(qa_1, \dots, qa_r, 1/a_1, \dots, 1/a_r; q)_\infty} {}_r\psi_r(a_1, \dots, a_r; b_1, \dots, b_r; q, x) \\ &= \frac{a_1^{r-1} (q, qa_1/a_2, \dots, qa_1/a_r, b_1/a_1, \dots, b_r/a_1, a_1x, q/a_1x; q)_\infty}{(qa_1, 1/a_1, a_1/a_2, \dots, a_1/a_r, qa_2/a_1, \dots, qa_r/a_1; q)_\infty} \\ & \times {}_r\varphi_{r-1} \left(qa_1/b_1, \dots, qa_1/b_r; qa_1/a_2, \dots, qa_1/a_r; q, \frac{b_1 \cdots b_r}{a_1 \cdots a_r x} \right) \\ & + \text{idem}(a_1; a_2, \dots, a_r). \end{aligned}$$

The notation $\text{idem}(a_1; a_2, \dots, a_r)$ after an expression stands for the sum of the r expressions obtained from the preceding expression by interchanging a_1 with each a_k , $k = 2, 3, \dots, r$.

A special case of Slater's formula gives Ramanujan's summation formula.

Remark 1. *If we put $r = 1$ in theorem 2, we obtain Ramanujan's sum for ${}_1\psi_1(a; b; q, x)$ [7, page 57]*

$$\begin{aligned} {}_1\psi_1(a; b; q, z) &= \frac{(q, b/a, az, q/az; q)_\infty}{(b, q/a, z, b/az; q)_\infty} \\ &= \frac{(b/a, q; q)_\infty}{(b, q/a; q)_\infty} \frac{\theta(-az)}{\theta(-z)} {}_1\varphi_0 \left(a; -; q, \frac{q}{az} \right). \end{aligned}$$

We put $r = 2$ and take the limit $b_2 \rightarrow 0$ in theorem 2, we obtain the following corollary:

Corollary 1. *For any $0 < |x| < 1$, we have*

$$\begin{aligned} {}_2\psi_2(a_1, a_2; b_1, 0; q, x) &= \frac{(qa_1, qa_2, 1/a_2, qa_1/a_2, b_1/a_1, q; q)_\infty}{(b_1, q/a_1, q/a_2, qa_1, a_1/a_2, qa_2/a_1; q)_\infty} \\ & \times \frac{\theta\left(-\frac{a_1x}{q}\right)}{\theta\left(-\frac{x}{q}\right)} {}_1\varphi_1 \left(\frac{qa_1}{b_1}; \frac{qa_1}{a_2}; q, \frac{qb_1}{a_2x} \right) + \text{idem}(a_1; a_2). \end{aligned}$$

3.2 Proof of main theorem

In this subsection, we give the proof of the main theorem by the using of the q -Borel-Laplace transformations.

Proof. We apply the q -Borel transformation to the divergent series (1). Then, we obtain the following expression for the Borel transform of (1) by the using of corollary 1:

$$\begin{aligned}
& (\mathcal{B}_q^+ {}_2\psi_1(a_1, a_2; b_1; q, x))(\xi) = {}_2\psi_2(a_1, a_2; b_1, 0; q, -\xi) \\
& = \frac{(qa_1, qa_2, 1/a_2, qa_1/a_2, b_1/a_1, q; q)_\infty}{(b_1, q/a_1, q/a_2, qa_1, a_1/a_2, qa_2/a_1; q)_\infty} \frac{\theta\left(\frac{a_1\xi}{q}\right)}{\theta\left(\frac{\xi}{q}\right)} \\
& \times {}_1\varphi_1\left(\frac{qa_1}{b_1}; \frac{qa_1}{a_2}; q, -\frac{qb_1}{a_2\xi}\right) + \text{idem}(a_1; a_2) \\
& =: \psi(\xi).
\end{aligned}$$

We also apply the q -Laplace transformation to the function $\psi(\xi)$ as follows:

$$\begin{aligned}
& (\mathcal{L}_{q,\lambda}^+ \psi(\xi))(x) = \sum_{n \in \mathbb{Z}} \frac{\psi(\lambda q^n)}{\theta(\lambda q^n/x)} \\
& = \frac{(qa_1, qa_2, 1/a_2, qa_1/a_2, b_1/a_1, q; q)_\infty}{(b_1, q/a_1, q/a_2, qa_1, a_1/a_2, qa_2/a_1; q)_\infty} \\
& \times \sum_{n \in \mathbb{Z}} \frac{\left(\frac{\lambda}{x}\right)^n q^{\frac{n(n-1)}{2}} \theta\left(\frac{a_1\lambda}{q} q^n\right)}{\theta\left(\frac{\lambda}{x}\right) \theta\left(\frac{\lambda}{q} q^n\right)} \sum_{k \geq 0} \frac{(qa_1/b_1; q)_k \left\{(-1)^k q^{\frac{k(k-1)}{2}}\right\}}{(qa_1/a_2)_k (q; q)_k} \left(-\frac{qb_1}{a_2\lambda} q^{-n}\right)^k \\
& + \text{idem}(a_1; a_2) \\
& = \frac{(qa_1, qa_2, 1/a_2, qa_1/a_2, b_1/a_1, q; q)_\infty}{(b_1, q/a_1, q/a_2, qa_1, a_1/a_2, qa_2/a_1; q)_\infty} \frac{\theta(a_1\lambda/q)}{\theta(\lambda/q)} \frac{\theta(\lambda/a_1x)}{\theta(\lambda/qx)} \\
& \times {}_2\varphi_1(qa_1/b_1, 0; qa_1/a_2; q, b_1/a_1a_2x) + \text{idem}(a_1; a_2).
\end{aligned}$$

Therefore, we obtain the conclusion. \square

Remark 2. We set the functions

$$C_1(x) := \frac{(1/a_2, qa_1/a_2, b_1/a_1, q; q)_\infty}{(b_1, q/a_1, a_1/a_2, qa_2/a_1; q)_\infty} \frac{\theta(a_1\lambda/q)}{\theta(\lambda/q)} \frac{\theta(a_1qx/\lambda)}{\theta(qx/\lambda)} \frac{\theta(x)}{\theta(a_1x)}$$

and

$$C_2(x) := \frac{(1/a_1, qa_2/a_1, b_1/a_2, q; q)_\infty}{(b_1, q/a_2, a_2/a_1, qa_1/a_2; q)_\infty} \frac{\theta(a_2\lambda/q)}{\theta(\lambda/q)} \frac{\theta(a_2qx/\lambda)}{\theta(qx/\lambda)} \frac{\theta(x)}{\theta(a_2x)},$$

the new connection formula can be rewritten in the following form:

$$(\mathcal{L}_{q,\lambda}^+ \circ \mathcal{B}_q^+ {}_2\psi_1(a_1, a_2; b_1; q, x))(x) = C_1(x)v_1(x) + C_2(x)v_2(x).$$

These connection coefficients $C_1(x)$ and $C_2(x)$ are q -elliptic functions.

Acknowledgements

The author would like to give heartfelt thanks to Professor Yousuke Ohyama who provided carefully considered feedback and many valuable comments.

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